Directional Derivatives

1. Def:
Let $f: U \subset \mathbb{R}^n \to \mathbb{R}$. The directional derivative of f at a in direction $\vec{v}$, denoted by $D_v(f(a))$, is given by

1. $\lim_{t \to 0} \frac{f(a+t\vec{v}) - f(a)}{t\|\vec{v}\|}$

2. $\nabla f(a) \cdot \frac{\vec{v}}{\|\vec{v}\|}$

Note: If $\nabla f(x) \neq 0$, then $\nabla f(x)$ points in the direction along which f is increasing the fastest.

2. Rate of Change:

- The rate of change of f at a in the direction of $\vec{v}$, where $\vec{v}$ MUST be a unit vector is given by

1. $\lim_{t \to 0} \frac{f(a+t\vec{v}) - f(a)}{t\|\vec{v}\|}$

2. $\nabla f(a) \cdot \frac{\vec{v}}{\|\vec{v}\|}$

– Note: If $\vec{v}$ is a unit vector, then the rate of change of f in direction $\vec{v}$ is given by $\nabla f(x) \cdot \vec{v} = \|\nabla f(x)\| \cos\theta$, where θ is the angle between $\vec{v}$ and $\nabla f(x)$. This is maximum when $\theta = 0$; that is, when ∇f and $\vec{v}$ are parallel. If $\nabla f(x) = 0$, then the rate of change is 0 for any $\vec{v}$. $\vec{v} = [\cos\theta, \sin\theta]$.

3. Examples:

1. Let $f(x,y,z) = x^2 e^{-yz}$. Find the rate of change of f in the direction of $\vec{v} = \left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]$ at the point $(1,0,0)$.

Soln:
$\vec{v}$ is already a normal vector.

$$\nabla f(1,0,0) = [2xe^{-yz}, -x^2ze^{-yz}, -x^2ye^{-yz}]\Big|_{(1,0,0)}$$
$$= [2,0,0]$$

$$[2,0,0] \cdot \left[\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right] = \frac{2}{\sqrt{3}}$$

2. Using the same function and point as in 1, but let $\vec{v} = [1,1,1]$. Find the r.o.c.

Soln:
Since $\vec{v}$ is no longer a normal vector, we must normalize it first.

$$\vec{w} = \frac{\vec{v}}{\|\vec{v}\|}$$
$$= \left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]$$

Now, $\vec{w}$ is a normal vector, and we can find the rate of change.

The r.o.c. is $\frac{2}{\sqrt{3}}$.

3. Let $f(x,y) = e^{xy}\sin(x+y)$.
   a) In what direction(s) is f increasing the fastest, starting at $(0, \frac{\pi}{2})$.

Soln:
Recall that $\nabla f(x)$ points in the direction along which f is increasing the fastest.

$$\nabla f = [ye^{xy}\sin(x+y) + e^{xy}\cos(x+y),$$
$$xe^{xy}\sin(x+y) + e^{xy}\cos(x+y)]$$

$$\nabla f(0, \tfrac{\pi}{2}) = [\tfrac{\pi}{2}, 0]$$

$\therefore$ f is increasing fastest in the direction of $[\frac{\pi}{2}, 0]$.

b) In what direction(s), starting at $(0, \frac{\pi}{2})$ is f changing at 50% of its max rate?

Soln:

$\nabla f(x) \cdot \vec{v} = \|\nabla f(x)\| \cdot \cos\theta$

$\nabla f(0, \frac{\pi}{2}) = [\frac{\pi}{2}, 0]$

$\|\nabla f(0, \frac{\pi}{2})\| = \frac{\pi}{2}$

$$\nabla f(0, \tfrac{\pi}{2}) \cdot \vec{v} = (\tfrac{\pi}{2})(\tfrac{1}{2})$$
$$= \tfrac{\pi}{4}$$

$$\nabla f(0, \tfrac{\pi}{2}) \cdot \vec{v} = \|\nabla f(0, \tfrac{\pi}{2})\| \cdot \cos\theta$$
$$\tfrac{\pi}{4} = \tfrac{\pi}{2}\cos\theta$$
$$\tfrac{1}{2} = \cos\theta$$
$$\theta = \pm \tfrac{\pi}{3}$$
$$\sin(\tfrac{\pi}{3}) = \sqrt{3}/2$$
$$\sin(-\tfrac{\pi}{3}) = -\sqrt{3}/2$$

$$\therefore \vec{v} = [\tfrac{1}{2}, \sqrt{3}/2] \text{ and } [\tfrac{1}{2}, -\sqrt{3}/2]$$